8. EXERCISE SHEET, RETURN DATE JULY 2./3.

Exercise 1

Prove the fourth order difference formula for the second derivative

$$u''(x) - \frac{-30u(x) + 16u(x+h) + 16u(x-h) - u(x-2h) - u(x+2h)}{12h^2} \in O(h^4)$$

Hint: Taylor expansions around x.

Exercise 2

We want to solve the equation

$$u''(x) = 1,\ u(0) = u(1) = 0$$

numerically using finite differences. We use the equidistant grid

$$x_i = \frac{i}{2^d},\ i = 1, ..., 2^d - 1$$

Discretizing the differential equations with the second order difference quotient $\partial_j^{-h}\partial_j^{+h}$ on this grid we arrive at a system of linear equations with a coefficient matrix K_d. Write a function in C that generates the coefficient matrix K_d.

Exercise 3

Solve the system

$$K_d u_d = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

using your gradient descent algorithm and your cg algorithm. Use the 0-vector as initial guess. In both cases plot the individual iteration steps of the gradient methods such that you can go forward and backward using two keys. Which method shows better convergence?

Exercise 4

Calculate the eigenvalues and condition number κ of K_n. Hint: scale the Chebychev polynomials of the second kind appropriately. Their roots are

$$a_k^n = cos\left(\frac{k\pi}{n+1}\right),\ k = 1, ..., n.$$

This condition number is of interest, because we can bound the convergence of the cg algorithm in terms of κ. For $d = 1$ the condition number is obviously 1, for $d = 2$ it is also easily calculated by hand and is 5.82...

Write a function in C that returns the condition number of K_n for the input variable n.

In the following homework sheet you will use a preconditioning technique to reduce this condition number and thus increase the convergence rate of the cg algorithm.

Exercise 5 (optional)

Use a sparse matrix format for all the computations.

Exercise 6

Start preparing for your oral exams.